Problem Set 4

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 4 covers materials from §3.3 – §4.1.

1. Find the derivative of each of the following functions:
   (a) $f(x) = \sqrt{(x^2+5)(2x+3)}$
   (b) $g(x) = \sqrt{x(ex^e + e^e)}$
   (c) $h(x) = \sin\sqrt{x} + \sqrt{\cos x} + \sqrt{\sin\sqrt{x}}$
   (d) $p(x) = 3^{\cos x} e^{x-\sin(x^2)}$
2. Using the definition of the Euler number e (Definition 3.32), prove that
$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1.$$
3. Using $e = \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$ if necessary, evaluate each of the following limits if it exists.
   (a) $\displaystyle\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^{x+1}$
   (b) $\displaystyle\lim_{x\to 0}(1+\tan x)^{\cot x}$
   (c) $\displaystyle\lim_{x\to+\infty}\left(1+\frac{3}{x}+\frac{2}{x^2}\right)^{x}$
   (d) $\displaystyle\lim_{x\to+\infty}\left(\frac{x^2-2x-3}{x^2-3x-2}\right)^{x}$
4. Suppose that an amount $\$A_0$ is invested in a nominal annual interest rate $r \times 100\%$. If the interest is compounded annually (1 time per year), then the total amount $\$A$ after t years is
$$A(t) = A_0(1+r)^t.$$
   (a) If the interest is compounded every three months (4 times per year), what is the total amount after t years? What if it is compounded every month (12 times per year)?
   (b) In finance, we say that the interest is ***compounded continuously*** if it is compounded n times per year and $n \to +\infty$. Find the formula for the total amount $\$A$ of the investment after t years if the interest is compounded continuously. Specifically, if the initial investment is $\$10000$ and the nominal annual interest rate is 6%, what is the total amount after 10 years?
5. In Problem Set 1, we have defined the ***hyperbolic functions*** $\sinh$, $\cosh$ and $\tanh$ as
$$\sinh x \coloneqq \frac{e^x - e^{-x}}{2}, \qquad \cosh x \coloneqq \frac{e^x + e^{-x}}{2}, \qquad \tanh x \coloneqq \frac{\sinh x}{\cosh x}$$
respectively. Now we define three more hyperbolic functions $\coth$, sech and csch by
$$\coth x \coloneqq \frac{1}{\tanh x}, \qquad \operatorname{sech} x \coloneqq \frac{1}{\cosh x}, \qquad \operatorname{csch} x \coloneqq \frac{1}{\sinh x}$$
respectively.

Using the rules of differentiation and the fact that

$$\frac{d}{dx}e^x = e^x,$$

prove the following differentiation formulas for these six hyperbolic functions:

(a) $\dfrac{d}{dx}\sinh x = \cosh x$

(b) $\dfrac{d}{dx}\cosh x = \sinh x$

(c) $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$

(d) $\dfrac{d}{dx}\coth x = -\operatorname{csch}^2 x$

(e) $\dfrac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$

(f) $\dfrac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$

6. In economics, the quantity demanded Q of a good in general decreases when its price P increases. The ***price elasticity of demand*** E_d of a good, defined by

$$E_d = \frac{P}{Q}\frac{dQ}{dP},$$

measures the rate of percentage change in quantity demanded of the good with respect to a one percent increase in its price. For each of the following relationships between P and Q, compute the price elasticity of demand E_d as a function of P:

(a) $PQ = 1013$

(b) $Q = 1013e^{-P}$

(c) $Q = 1013 - P^a$, where $a > 0$ is a constant

7. Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \dfrac{\sin 2x + \sin x}{e^x - a} & \text{if } x < 0 \\ be^x - x & \text{if } x \geq 0 \end{cases},$$

where a, b are non-zero real numbers. If f is continuous, find the values of a and b.

8. Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} e^{2x} & \text{if } x \geq 0 \\ \sin 2x \cos 2x & \text{if } x < 0 \end{cases}.$$

Compute the derivative of f at 0, i.e. compute $f'(0)$.

9. The following table shows some values of the functions f, g, f' and g'.

x	-1	0	1	2
$f(x)$	1	2	3	-2
$g(x)$	-2	1	7	4
$f'(x)$	2	-4	-3	-1
$g'(x)$	3	0	1	-5

Using the information from this table, find

$$\frac{d}{dx}\left[f\left(\sqrt[3]{g(x)}\right)f(x)\right]^2\Big|_{x=0} \quad \text{and} \quad \frac{d}{dx}\frac{g([f(x)]^2+1)}{f\big(f(x)\big)}\Big|_{x=-1}.$$

10. Let a be a positive number. Show that at each point on the curve

$$\sqrt{x}+\sqrt{y}=\sqrt{a}$$

except the end-points $(a,0)$ and $(0,a)$, the sum of the x- and y-intercepts of the tangent line to the curve is always equal to a.

11. For each of the following, find $\frac{dy}{dx}$ in terms of x and y.

(a) $xy^2+e^y=\cos(x+y^2)$

(b) $x^2+y^2-\arcsin y=0$

(c) $\dfrac{1}{2}\ln(x^2+y^2)+\arctan\dfrac{y}{x}=1$

12. Find the derivative of each of the following functions f.

(a) $f(x)=2^x(x\sin x+\cos x)$

(b) $f(x)=\sqrt{\dfrac{1+x+x^2}{1-x+2x^2}}$

(c) $f(x)=\sqrt{(x\sin x)\sqrt{1-e^{-x}}}$

(d) $f(x)=(\ln x)^{\frac{1}{\ln x}}$

13. Find the derivative of each of the following functions for $x>0$.

(a) $f(x)=e^{e^e}$

(b) $f(x)=x^{e^e}$

(c) $f(x)=e^{x^e}$

(d) $f(x)=e^{e^x}$

(e) $f(x)=x^{x^e}$

(f) $f(x)=x^{e^x}$

(g) $f(x)=e^{x^x}$

(h) $f(x)=x^{x^x}$

14. Show that the derivatives of the following functions all have the same formula.

(a) $f(x)=\dfrac{1}{2}\ln\left(\dfrac{1+\sin x}{1-\sin x}\right)$

(b) $f(x)=\ln(\sec x+\tan x)$

(c) $f(x)=\ln\left(\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)\right)$

15. By considering the differentiation formulas

$$\frac{d}{dx}\tan x=\sec^2 x \qquad \text{and} \qquad \frac{d}{dx}\sec x=\sec x\tan x\,,$$

as well as the derivative of the inverse of a function, prove that

$$\frac{d}{dx}\arctan x=\frac{1}{1+x^2} \qquad \text{and} \qquad \frac{d}{dx}\operatorname{arcsec} x=\frac{1}{|x|\sqrt{x^2-1}}.$$

16. The following table shows some values of the functions f, g, f', g', f'' and g''.

x	-2	0	1	3
$f(x)$	-5	-2	0	2
$g(x)$	3	2	1	0
$f'(x)$	5	4	3	1
$g'(x)$	0	-3	2	-2
$f''(x)$	-1	0	9	5
$g''(x)$	-1	3	0	-6

Using the information from this table, find the derivatives

$$\frac{d}{dx}[(f^{-1})'(x)(g')^{-1}(x)]\Big|_{x=0} \qquad \text{and} \qquad \frac{d}{dx}(g\circ f')^{-1}\big(f^{-1}(x)\big)\Big|_{x=-2},$$

assuming that all the relevant inverses exist and are differentiable.

17. For each positive integer n, find the n^{th} derivative of each of the following functions.
    (a) $f(x) = e^{3x}$
    (b) $f(x) = \ln x$
    (c) $f(x) = \sin ax$, where $a \in \mathbb{R}$ is a constant

18. Find $\frac{d^2y}{dx^2}$ in terms of x and y if $\sqrt{y} + xy = 1$.

19. In physics, Newton's Second Law states that the net force F acting on an object with mass m and velocity v is equal to the "rate of change of momentum", i.e.

$$F = \frac{d}{dt}(mv),$$

where t is time. (If the mass m is constant, then this is just $F = ma$ where $a = dv/dt$ is the acceleration of the object.) Now suppose that the mass of the object also changes with its velocity according to the relation

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where m_0 and c are positive constants; m_0 is the mass of the object at rest and c is the speed of light. Show that in this situation, the net force acting on the object is

$$F = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}.$$

20. An object moves along a line modeled by the x-axis. Its position (i.e., its x-coordinate) after t seconds is given by

$$x(t) = a\sin t + b\cos t,$$

where a and b are positive constants.
    (a) If $a = \sqrt{3}$ and $b = 1$, find the velocity and acceleration of this object when $t = \pi/6$.
    (b) Show that the sum of the squares of the position and the velocity of the object is always constant.

21. In finance, the market price V of a bond investment is often expressed as a function of the yield r (i.e. the interest rate). The "risk" of this investment is often measured in the form of the percentage sensitivity of V to changes in r, using the (***modified***) ***duration*** D and the (***modified***) ***convexity*** C of the bond, defined by

$$D = -\frac{1}{V}\frac{dV}{dr} \qquad \text{and} \qquad C = \frac{1}{V}\frac{d^2V}{dr^2}$$

respectively.

(a) Calculate the duration and the convexity of the bonds with the following price functions, when $r = 0.06$.

(i) $V = 1000e^{-10r}$

(ii) $V = \dfrac{1000}{(1+r)^{10}}$

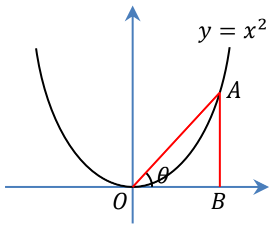
(iii) $V = \dfrac{250}{(1+r)^5} + \dfrac{1250}{(1+r)^{10}}$

(b) Show that the convexity of a bond is related to its duration by

$$C = D^2 - \frac{dD}{dr}.$$

22. A spherical capsule is melted in water. Assuming that in the process of melting, the capsule remains spherical. It is given that the volume of the capsule is decreasing at a rate of $9\pi\ \mathrm{cm}^3\ \mathrm{s}^{-1}$. Find the rates of change of its radius and surface area when its radius is 4 cm.

23. Consider a line in the coordinate plane that passes through the origin O and has an inclination angle $\theta \in (0, \pi/2)$. This line meets the parabola $y = x^2$ at the points O and A. Another vertical line through A intersects the x-axis at the point B, as shown in the diagram below.



Now suppose that the angle θ is increasing at a rate of $\frac{1}{3}$ radians per minute.

(a) When $\theta = \pi/3$, find the rate of change of the area of ΔOBA.

(b) When the length OA is $\sqrt{2}$ units, find the rate of change of the length of OA.